LEC 17. By Fourier transform, we get

$$\hat{u}(w,t) = c(w)\, e^{-kw^2 t}$$

directly without using the separation of variables.

Previously we have shown the inverse Fourier transform of $\hat{u}$. Instead, we have another simpler way to get the same result.

Convolution theorem

Note that $\hat{u}(w,t)$ is the product of two transforms of some function. ($c(w) = \hat{f}$. and e^{-kw^2t} is the F. transform of $\sqrt{\pi/kt}\, e^{-x^2/(4kt)}$).

Suppose ~~F(w) and~~ $\hat{f}(w)$ and $\hat{g}(w)$ are the Fourier transforms of $f(x)$ and $g(x)$.

$$\hat{f}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{iwx}\, dx \qquad \hat{g} = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\, e^{iwx}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(w)\, e^{-iwx}\, dx \qquad g(x) = \int_{-\infty}^{\infty} \hat{g}(w)\, e^{-iwx}\, dx$$

We want to determine the function $h(x)$ where $\hat{h} = \hat{f}\hat{g}$.

$$h(x) = \int_{-\infty}^{\infty} \hat{h}(w)\, e^{-iwx}\, dx = \int_{-\infty}^{\infty} \hat{f}(w)\,\hat{g}(w)\, e^{-iwx}\, dw$$

(we substitute either $\hat{f}$ or $\hat{g}$)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)\left(\int_{-\infty}^{\infty} g(y)\, e^{iwy}\, dy\right) e^{-iwx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} \hat{f}(w)\, e^{-iw(x-y)}\, dw\right) dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy. \qquad (*)$$

The integral in (*) is called the convolution of $f(x)$ and $g(x)$.

Hence the inverse Fourier transform of the product of two Fourier transforms is $1/2\pi$ times the convolution of the two functions.

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy = f*g$$

By change of variables

$$x-y=w \qquad dy = -dw$$

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-w)\, g(w)\, dw = g*f.$$

Hence convolution is commutative, i.e. $f*g = g*f$.

Heat equation

By convolution theorem.

$$u(x,t) = f(x) * \left(\sqrt{\frac{\pi}{kt}}\, e^{-x^2/4kt}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) \sqrt{\frac{\pi}{kt}}\, e^{-(x-y)^2/4kt}\, dy.$$

Summary of ~~sol~~ solving PDE by Fourier transform.

1. Take the (spatial) Fourier transform of PDE
2. solve the obtained ODE
3. Apply IC.
4. Use convolution theorem to get the solution.

Parseval's identity.

By convolution theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y) f(x-y)\, dy = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)\, e^{-iwx}\, dw.$$

Take $x=0$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y) f(-y) = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)\, dw$$

If we pick $g^*(x) = f(-x)$, where $*$ denotes the ~~convolution theorem con~~ complex conjugate

Note:

$$\hat{f}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{iwx}\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(-s) e^{-iws}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(x) e^{-iwx}\, dx = G^*(w).$$

Hence

$$\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{g(y) g^*(y)}_{|g(x)|^2}\, dy = \int_{-\infty}^{\infty} \underbrace{G^*(w) G(w)}_{|G(w)|^2 \text{: energy per wave number / spectral energy}}\, dw.}_{\text{total energy}}$$

Examples

1-D Wave equation on an infinite interval.

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty \\ u(x,0) = f(x) & u_t(x,0) = 0 \end{cases}$$

Take the spatial F.T.

$$\hat{u}_{tt} = -c^2 w^2 \hat{u}$$

$$\text{IC:} \quad \hat{u}(w,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{iwx}\, dx$$

$$\hat{u}_t = 0.$$

General sol to ODE $\hat{u}(w,t) = A(w)\cos(cwt) + B(w)\sin(cwt)$.

IC $\Rightarrow$ $B(w) = 0$.

$$A(w) = \hat{u}(w,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{iwx}\, dx$$

Now by inverse F.T.

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(w,0)\cos(cwt)\, e^{-iwx}\, dx.$$

By Euler's formula.

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{u}(w,0)\left(e^{-iw(x-ct)} + e^{-iw(x+ct)}\right) dw$$

since $f(x) = \int_{-\infty}^{\infty} \hat{u}(w,0)\, e^{-iwx}\, dx$

hence

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right].$$

Note: $u(x,t)$ is a sum of two travelling waves.

Ex. 2 Laplace equation in a half plane.

Suppose the temperature is specified on a infinite wall $y=0$. Consider

$$\begin{cases} \Delta u = u_{xx} + u_{yy} = 0 & y > 0 \\ u(x,0) = f(x) & y = 0. \end{cases}$$

Assume $u \to 0$ as either $x \to \pm\infty$, $y \to \infty$.

Note: we have two homogeneous BC : $x \to \pm\infty$, $u \to 0$.

Taking the spatial F.T. in x.

$$\hat{u}_{yy} - w^2 \hat{u} = 0.$$

Since $u(x,y) \to 0$ as $y \to +\infty$, $\hat{u}(w,y) \to 0$ as $y \to \infty$.

At $y=0$ $\quad \hat{u}(w,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{iwx} dx.$

General solution $\hat{u}(w,y) = a(w) e^{wy} + b(w) e^{-wy}$

Since ~~both~~ w could be either positive or negative,

$$\hat{u}(w,y) = c(w) e^{-|w|y}.$$

Also, $c(w) = \hat{f}(w)$.

Convolution theorem

$$\text{Let} \quad g(x,y) = \int_{-\infty}^{\infty} e^{-|w|y} e^{-iwx} dw$$

$$= \int_{-\infty}^{0} e^{wy} e^{-iwx} dw + \int_{0}^{\infty} e^{-wy} e^{-iwx} dy$$

$$= \left(\frac{e^{w(y-ix)}}{y-ix} \right) \Big|_{-\infty}^{0} + \left(\frac{e^{-w(y+ix)}}{-(y+ix)} \right) \Big|_{0}^{\infty}$$

$$= \frac{1}{y-ix} + \frac{1}{y+ix} = \frac{2y}{x^2+y^2}$$

Hence, the solution to Laplace equation in semi-infinite space $(y>0)$ subject to $u(x,0) = f(x)$ is

$$u(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) \frac{2y}{(x-z)^2+y^2} \, dz$$

Note: The solution is derived by assuming $f(x) \to 0$ as ~~$x \pm \infty$~~ $x \to \pm\infty$. It holds in a more general case ~~such that~~ once the integral is convergent.